Quantum vortices in systems obeying a generalized exclusion principle

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The paper deals with a planar particle system obeying a generalized exclusion principle (EP) and governed, in the mean field approximation, by a nonlinear Schrödinger equation. We show that the EP involves a mathematically simple and physically transparent mechanism, which allows the genesis of quantum vortices in the system. We obtain in a closed form the shape of the vortices and investigate its main physical properties.

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In literature experimental evidence has been presented of formation of quantized vortex structures in different macroscopic quantum systems. Vortices have been observed both in Bose (Bose-Einstein condensate of alkali atom clouds) and in Fermi (^{3}He-A superfluids, heavy fermion superconductors UPt_3 and $U_{0.97}Th_{0.03}Be_{13}$) systems [1–6]. Actually, there exist different theoretical models admitting stationary solutions of vortex type [7–17]. It is apparent that the vortex type solution is imposed by the nonlinearities introduced in the particular model adopted. For instance, by considering a nonrelativistic matter system, one can construct different models starting from nonlinear Schrödinger equations (NLSEs) or from the gauged NLSEs in the frame of a Maxwell or Chern-Simons theory.

Principal goal of the present work is to show that there exists a mathematically simple and physically transparent mechanism, imposed by the generalized exclusion principle (EP), which allows the genesis of quantum vortices in planar nonrelativistic particle systems.

Since 1932, it was clear that the effects due to the statistics and imposed to a system of free fermions by the Pauli exclusion principle can be simulated by a repulsive potential in the coordinate space [18]. In the same way for bosons, an attractive potential can simulate the statistical behavior of the system. After 1940, in different works, particle systems obeying to statistics, which are different from the standard Bose-Einstein and Fermi-Dirac ones, have been considered [19–22].

Recently, it has been studied a many body quantum system obeying a generalized exclusion-inclusion principle (EIP) [23–28]. The peculiar property of this system is that the EIP introduces an attractive or repulsive potential in the coordinate space and can simulate the intermediate (between bosonic and fermionic) behavior of the system. We consider some examples of real physical systems where EIP can be usefully applied. The Bose-Einstein condensation originates from an attraction of statistical nature (Bose-Einstein statistics) among the particles. In several systems the Bose-Einstein condensation is studied by means of a cubic NLSE that describes in mean field approximation an attractive interaction between two bodies. In place of the cubic and simplest interaction, other interactions can be considered as, for instance, the one introduced by EIP to simulate an attraction among the particles. Analogously, in the case of superfluids or semiconductors Fermionic systems, the repulsive interaction among the particles originated from the Fermi-Dirac statistics can be simulated by the EIP with $\kappa<0$. In condensed matter the motion of the couple electron hole in a semiconductor can be described again by EIP. In fact, while electrons and holes are fermions, and together can be considered as excited states behaving differently from a fermion or a boson. Moreover, in nuclear physics the interaction among the fermionic valence nucleons outside the core produces pairs of correlated nucleons that can be approximated as particles with a behavior intermediate between fermionic and bosonic ones. This nuclear state (quasideuteron state) can be viewed as a particle system that obeys to EIP.

The dynamics of the canonical quantum system obeying the EIP is governed by the following NLSE:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+W(\rho,\boldsymbol{j})\psi+i\mathcal{W}(\rho,\boldsymbol{j})\psi+V\psi, \tag{1}$$

where the real and imaginary parts of the nonlinearity are given respectively by:

$$W(\rho,\boldsymbol{j})=\kappa\frac{m}{\rho}\left(\frac{\boldsymbol{j}}{1+\kappa\rho}\right)^2, \tag{2}$$

$$\mathcal{W}(\rho,\boldsymbol{j})=-\kappa\frac{\hbar}{2\rho}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{j}\rho}{1+\kappa\rho}\right). \tag{3}$$

The free parameter κ is a constant that takes into account the intensity of the exclusion-inclusion statistical effects. It is easy to verify that the system described by Eq. (1) obeys the continuity equation

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0, \tag{4}$$

where $\rho=|\psi|^2$, while the quantum current $\boldsymbol{j}$ is given by

$$\boldsymbol{j}=-\frac{i\hbar}{2m}(1+\kappa\rho)(\psi^*\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\psi^*). \tag{5}$$

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Equation (4) assures the conservation of the particle number $N=\int \rho d^d x$ of the system.

We discuss now briefly the origin of the model described by Eq. (1). We start by considering the following nonlinear Fokker-Planck equation [23,24]:

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot\left[\frac{\boldsymbol{\nabla} S}{m}\rho(1+\kappa\rho)+D\boldsymbol{\nabla}\rho\right]=0, \tag{6}$$

being $\boldsymbol{\nabla} S/m$ the drift velocity and D the diffusion coefficient. In the case, $\kappa=0$ and $D=0$, the above Fokker-Planck equation reduces to the well-known continuity equation for ρ and the linear quantum mechanics can be obtained if we use the ansatz $\psi=\rho^{1/2}\exp(iS/\hbar)$. In Ref. [29], it has been considered the case $\kappa=0$, $D\neq 0$ and starting from Eq. (6) a new NLSE was obtained. Differently, starting from Eq. (6), after posing $\kappa\neq 0$ and $D=0$, Eq. (1) can be obtained.

The introduction in Eq. (1) of the factor $1+\kappa\rho$ originates from the presence of the EIP and allows us to take into account many particle quantum effects. In fact, the transition probability from the site $\boldsymbol{x}$ to $\boldsymbol{x}'$ is defined as $\pi(t,\boldsymbol{x}\rightarrow\boldsymbol{x}')=r(t,\boldsymbol{x},\boldsymbol{x}')\rho(t,\boldsymbol{x})[1+\kappa\rho(t,\boldsymbol{x}')]$ with $r(t,\boldsymbol{x},\boldsymbol{x}')$ the transition rate. The transition probability depends on the particle population $\rho(t,\boldsymbol{x})$ of the starting point $\boldsymbol{x}$, and also on the population $\rho(t,\boldsymbol{x}')$ of the arrival point $\boldsymbol{x}'$. For $\kappa\neq 0$ the EIP holds and the parameter κ quantifies how much the particle kinetics is affected by the particle population of the arrival point. If $\kappa>0$ the $\pi(t,\boldsymbol{x}\rightarrow\boldsymbol{x}')$ introduces an inclusion principle. In fact the population at the arrival point $\boldsymbol{x}'$ stimulates the transition, and the transition probability increases linearly with $\rho(t,\boldsymbol{x}')$. In the case $\kappa<0$ the $\pi(t,\boldsymbol{x}\rightarrow\boldsymbol{x}')$ takes into account the Pauli exclusion principle. If the arrival point $\boldsymbol{x}'$ is empty, $\rho(t,\boldsymbol{x}')=0$, and the $\pi(t,\boldsymbol{x}\rightarrow\boldsymbol{x}')$ depends only on the population of the starting point. If the arrival site is populated $0<\rho(t,\boldsymbol{x}')\leqslant\rho_{\max}$, and the transition is inhibited. The range of values the parameter κ can assume is bounded by the condition that $\pi(t,\boldsymbol{x}\rightarrow\boldsymbol{x}')$ be real and positive, as $r(t,\boldsymbol{x},\boldsymbol{x}')$ is. Thus we conclude that $\kappa\geqslant -1/\rho_{\max}$.

The form of the nonlinearity $\mathcal{W}(\rho,\boldsymbol{j})$ in Eq. (1) is imposed by the continuity equation (4), while the form of $W(\rho,\boldsymbol{j})$ is imposed by the requirement of the canonicity of the system. The Hamiltonian density of the system (1) is given by

$$\mathcal{H}=\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2+U_{\text{EIP}}+V\rho, \tag{7}$$

being

$$U_{\text{EIP}}=\kappa\rho^2\frac{(\boldsymbol{\nabla} S)^2}{2m}, \tag{8}$$

the nonlinear potential introduced by the EIP. In Ref. [27] it has been shown that the system (1) admits one-dimensional solitons.

Following the standard procedure of the linear quantum mechanics, we define the quantum velocity $\boldsymbol{v}$ through $\boldsymbol{j}=\boldsymbol{v}\rho$. Taking into account Eq. (5) and writing ψ in terms of the hydrodynamic variables $\psi=\rho^{1/2}\exp(iS/\hbar)$, we have

$$\boldsymbol{v}=(1+\kappa\rho)\frac{\boldsymbol{\nabla} S}{m}. \tag{9}$$

The quantum velocity can be expressed in terms of the Clebsch potentials S,λ,μ through $m\boldsymbol{v}=\boldsymbol{\nabla} S+\lambda\boldsymbol{\nabla}\mu$. It results that the EIP imposes the choice $\lambda=\kappa\rho$ and $\mu=S$ for the Clebsch potentials. Now, we define the vorticity $\boldsymbol{\omega}$ of the system, through $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{v}$ and obtain

$$\boldsymbol{\omega}=\frac{\kappa}{m}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla} S. \tag{10}$$

From Eq. (10), it results that Eq. (1) describes a vorticose system. In the present contribution, we will consider the planar, static vortex solutions of Eq. (1) in the case $\kappa=-\xi<0$, when EIP is reduced to an EP. By introducing the polar coordinates $r=\sqrt{x^2+y^2}$ and $\theta=\arctan(y/x)$ in the plane, we search for solutions of Eq. (1) in which $S=S(\theta)$ and $\rho=\rho(r)$. The continuity equation imposes $S=\hbar n\theta$ and consequently we write ψ as

$$\psi(r,\theta)=\rho(r)^{1/2}e^{in\theta}. \tag{11}$$

The parameter n must be integer in order to make ψ, given by Eq. (11), a single value function. In the following, we impose $\int\rho d^2x=N$, and therefore $\rho(\infty)=0$. Let us note that Eq. (1) is not Galilei invariant [28] so that traveling solutions cannot be obtained by boosting static solutions of Eq. (1).

The quantum velocity $\boldsymbol{v}$ for the vortices (11) becomes

$$\boldsymbol{v}=\frac{\hbar}{m}\frac{n}{r}(1-\xi\rho)\hat{\boldsymbol{e}}_\theta, \tag{12}$$

where $\hat{\boldsymbol{e}}_\theta$ is the unitary vector orthogonal to the position vector $\boldsymbol{r}=(x,y)$. After integration of Eq. (12) on the circle γ_∞ with center at the vortex core and with a radius $R\rightarrow\infty$, we obtain the following relevant property:

$$m\oint_{\gamma_\infty}\boldsymbol{v}\cdot d\boldsymbol{l}=2\pi\hbar n, \tag{13}$$

which justifies the name of vorticity index n. The vorticity $\boldsymbol{\omega}=\omega\hat{\boldsymbol{e}}_z$ of the system is given by

$$\omega=\frac{2\pi\hbar n}{m}\delta^2(\boldsymbol{r})-\xi\frac{\hbar}{m}\frac{n}{r}\frac{d\rho}{dr}, \tag{14}$$

and taking into account $\rho(\infty)=0$, it is easy to verify that the total vorticity depends exclusively on the behavior of the vortex core

$$\int \omega d^2x=2\pi n\frac{\hbar}{m}. \tag{15}$$

The definition of the vorticity $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{v}$, imposes the relation

$$\int \omega d^2x=\oint_{\gamma_\infty}\boldsymbol{v}\cdot d\boldsymbol{l}, \tag{16}$$

which can be easily verified by comparing Eqs. (13) and (15).

The angular momentum of the vortex, which is a vector orthogonal to the vortex plane, can be calculated as mean value of the operator $\hat{L}_z = -i\hbar\,\partial/\partial\theta$ and assumes the following quantized value:

$$L_z = nN\hbar, \tag{17}$$

which does not depend on the parameter ξ.

We can calculate now the complex nonlinearity $W + i\mathcal{W}$ in Eq. (1) for the vortex given by Eq. (11). The nonlinearity becomes a real one and Eq. (1) reduces to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi - \frac{\hbar^2}{m}\frac{\xi n^2}{r^2}\rho\psi + V\psi. \tag{18}$$

Equation (18) contains a nonlinearity very close to the one of cubic NLSE and describes a canonical quantum system with a Hamiltonian density given by Eq. (7), where the nonlinear potential introduced by the EP assumes the form

$$U_{\rm EP} = -\frac{\hbar^2}{2m}\frac{\xi n^2}{r^2}\rho^2. \tag{19}$$

As regards to the spatial shape of the vortex $\rho = \rho(r)$, we insert Eq. (11) into Eq. (18), and obtain the following second order ordinary differential equation:

$$\frac{1}{r\rho}\frac{d}{dr}\left(r\frac{d\rho}{dr}\right) - \frac{1}{2}\left(\frac{1}{\rho}\frac{d\rho}{dr}\right)^2 - \frac{2n^2}{r^2}(1-2\xi\rho) - \frac{4m}{\hbar^2}V(r) = 0. \tag{20}$$

In the following, we are interested to study the free vortices $V=0$. After introducing the dimensionless variable

$$z = 2n\log\frac{r}{r_n}, \tag{21}$$

where r_n is an arbitrary constant, Eq. (20) becomes

$$\frac{2}{\rho}\frac{d^2\rho}{dz^2} - \left(\frac{1}{\rho}\frac{d\rho}{dz}\right)^2 + 2\xi\rho - 1 = 0. \tag{22}$$

Now we consider the auxiliary function $y(\rho)$

$$y(\rho) = \left(\frac{1}{\rho}\frac{d\rho}{dz}\right)^2, \tag{23}$$

which, taking into account Eq. (22), obeys the differential equation

$$\frac{dy}{d\rho} + \frac{y}{\rho} - \frac{1}{\rho}(1-2\xi\rho) = 0. \tag{24}$$

The function $y(\rho)$ after integration of Eq. (24) assumes the form

$$y(\rho) = \frac{\alpha}{\rho} + 1 - \xi\rho, \tag{25}$$

being α an integration constant. Combining Eq. (23) and Eq. (25), we arrive at the following first order ordinary differential equation for the shape of the vortex:

$$\left(\frac{d\rho}{dz}\right)^2 = \alpha\rho + \rho^2(1-\xi\rho). \tag{26}$$

All the physical solutions of Eq. (26), whatever the value of the parameter $\alpha \in \mathbb{R}$, must be nonsingular, non-negative, and normalizable.

The Hamiltonian $H = \int \mathcal{H} d^2x$ of the system can be calculated starting from Eq. (7) and the expression (19) for $U_{\rm EP}$:

$$H = \frac{\hbar^2 n^2}{m}\int\left[\frac{\alpha}{2}\frac{1}{r^2} + \frac{1}{r^2}\rho(1-\xi\rho)\right]d^2x. \tag{27}$$

We must choose $\alpha = 0$ in order to have a finite value of the Hamiltonian. After integration on the variable θ, we obtain the following simple expression for the Hamiltonian:

$$H = 2\pi n\hbar\int_0^\infty j\,dr, \tag{28}$$

being $j = (\hbar n/mr)\rho(1-\xi\rho)$ the value of the current $\boldsymbol{j} = j\hat{\boldsymbol{e}}_\theta$ given by Eq. (5). Finally, also the differential equation for the vortex shape assumes a very simple form

$$\left(\frac{d\rho}{dz}\right)^2 = \rho^2(1-\xi\rho), \tag{29}$$

and can be easily integrated, obtaining the vortex profile in an explicit form

$$\rho(r) = \frac{4}{\xi}\left[\left(\frac{r}{r_n}\right)^n + \left(\frac{r_n}{r}\right)^n\right]^{-2}. \tag{30}$$

Therefore, the wave function of the vortex becomes

$$\psi(r,\ \theta) = \frac{2}{\sqrt{\xi}}\left[\left(\frac{r}{r_n}\right)^n + \left(\frac{r_n}{r}\right)^n\right]^{-1}\exp(in\theta). \tag{31}$$

The free parameter r_n, which absorbs the integration constant related to Eq. (29), can be calculated from the normalization condition $2\pi\int_0^\infty \rho(r) r\,dr = N$ and assumes the value

$$r_n = \frac{|n|}{2\pi}\sqrt{\xi N \sin\frac{\pi}{|n|}}, \tag{32}$$

where n is an integer number with $n \geqslant 2$ in order to have normalizable profiles for the vortices. The two vortex corresponds to the ground state of the system. The parameter r_n represents the distance from the vortex core to the point in which ρ assumes its maximum values: $\rho(r_n) = \xi^{-1}$. We have, $\forall r \geqslant 0$ that $0 \leqslant \rho \leqslant \xi^{-1}$, in agreement with the EP. From Eqs. (14) and (30) we obtain the following expression for the vorticity:

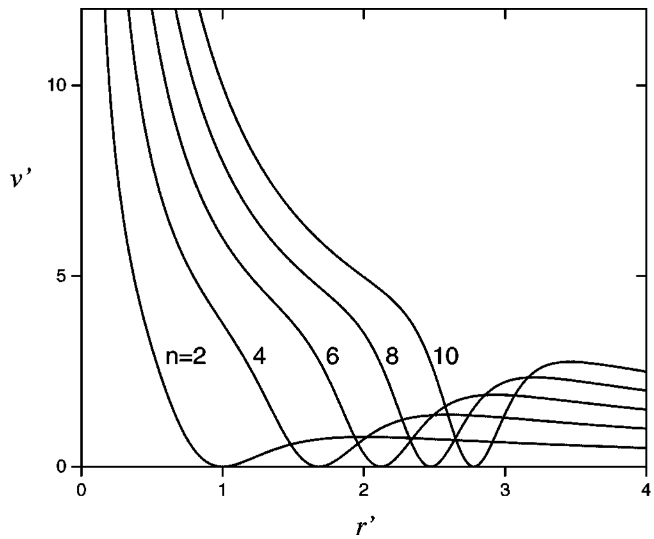

FIG. 1. Plot of the dimensionless quantum velocity $\nu' = \nu m r_2/\hbar$ with $r_2 = \sqrt{\xi N/\pi}$ versus the dimensionless distance from the vortex core $r' = r/r_2$ for the vortices with $n = 2,4,6,8,10$.

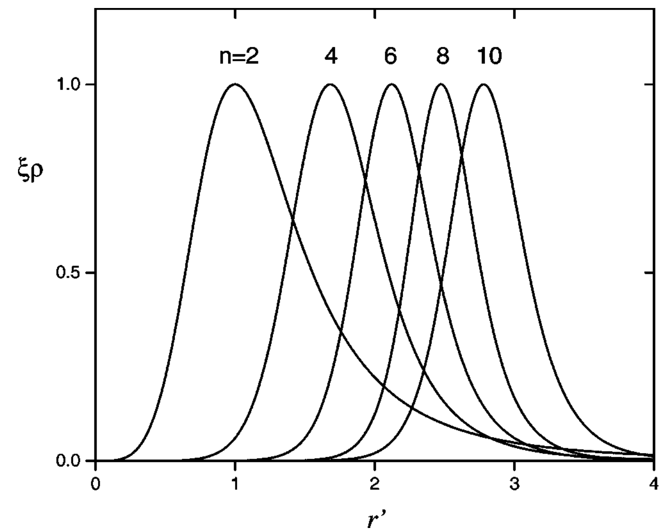

FIG. 3. Plot of the dimensionless profile $\xi\rho$ versus the dimensionless distance from the vortex core $r' = r/r_2$ with $r_2 = \sqrt{\xi N/\pi}$ for the vortices with $n = 2,4,6,8,10$.

$$\omega = \frac{2\pi\hbar n}{m}\,\delta^2(\boldsymbol{r}) - \frac{8n^2\hbar}{m r_n^2}\left(\frac{r}{r_n}\right)^{2(n-1)}\left[1 - \left(\frac{r}{r_n}\right)^{2n}\right] \times \left[1 + \left(\frac{r}{r_n}\right)^{2n}\right]^{-3}. \qquad (33)$$

Figure 1 shows the plot of the dimensionless quantum velocity $\nu' = \nu m r_2/\hbar$ with $r_2 = \sqrt{\xi N/\pi}$, obtained by combining Eqs. (12) and (30), versus the dimensionless distance from the vortex core $r' = r/r_2$ for the vortices with $n = 2,4,6,8,10$. We observe that $\boldsymbol{v}$ is equal to zero for $r = r_n$ where ρ reaches its maximum value. Figure 2 reports the behavior of the nonsingular part of the dimensionless vorticity $\omega' = \omega m r_2^2/\hbar$, given by Eq. (33) for the same vortices of Fig. 1. Figure 3 shows the dimensionless profile $\xi\rho$ versus r' for the same vortices of the previous figures. Finally, a 3D representation of the profile $\xi\rho$ for the ground vortex $n = 2$ in the 2D-dimensionless space: $(x' = x/r_2,\ y' = y/r_2)$ is reported in Figure 4.

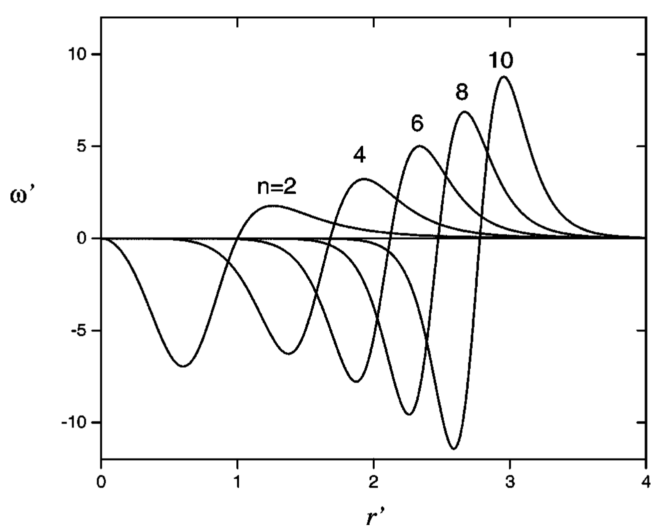

FIG. 2. Plot of the nonsingular part of the dimensionless vorticity $\omega' = \omega m r_2^2/\hbar$ with $r_2 = \sqrt{\xi N/\pi}$ versus the dimensionless distance from the vortex core $r' = r/r_2$ for the vortices with $n = 2,4,6,8,10$.

The energy $E = H$ of the n vortex can be calculated easily by substituting Eq. (30) into Eq. (28) and performing the integration

$$E = |n|\frac{\hbar^2 k^2}{2m}, \quad k^2 = \frac{8\pi}{3\xi}. \qquad (34)$$

The energy of the system results to be quantized and the energy spectrum lower bounded.

We recall that the family of the vortices (30) is related to the vortices $\rho_{JP}(r)$ of Ref. [12] through $\rho(r) \propto r^2 \rho_{JP}(r)$. We remark that the vortices $\rho_{JP}(r)$ are obtained as self-dual static solutions of a Chern-Simons model and correspond to the same energy state with energy equal to zero. On the

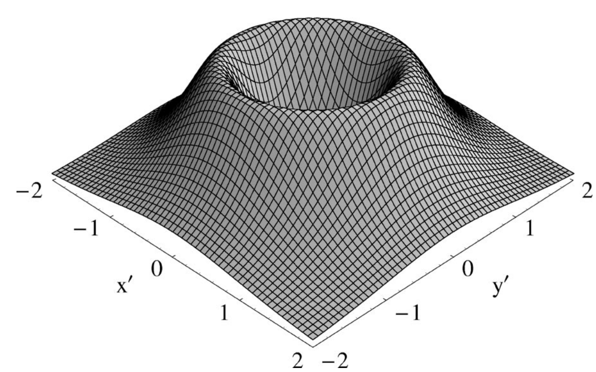

FIG. 4. 3D representation of the dimensionless profile $\xi\rho$ for the ground vortex $n = 2$ in the 2D-dimensionless space: $(x' = x/r_2,\ y' = y/r_2)$ with $r_2 = \sqrt{\xi N/\pi}$.

contrary, for the vortex family (30), we have that any vortex corresponds to a different state of the system whose energy is given by Eq. (34).

We conclude by noting that very recently [30], it has been considered a generalization of the present model (describing neutral particles that obey the EIP) in the cases of nonrelativistic charged particles. In this modified model the matter field obeying the EIP is minimally coupled to a gauge field whose dynamics are described within the frame of the Chern-Simons picture. The model is a canonical one and admits self-dual static nontopological vortex solutions with zero energy and linear momentum. The expressions of the main physical quantities associated to these solutions are obtained. The electric charge and the angular momentum are derived analytically, while the shape, together with the electric and magnetic fields of the vortex, are obtained numerically. This model can be considered as a continuous deformation of the Jackiw and Pi one [12], performed by the parameter κ that takes into account the EIP.

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